Dynamic Control of Water Levels in a Root Screening Facility

DLF-Trifolium Problem

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Abstract

Possibilities for modeling and control of moisture level for an underground irrigation system are investigated with the ultimate goal of designing a system for a root screening facility. To this purpose, several modeling approach are considered. In a two-phase flow approach for water and air flow in soil, the governing mass balance equations are solved numerically using pP formulation implemented in the OpenGeoSys FEM source code. In another approach where air pressure is assumed constant, the Richards model is used. The Richards flow model has been implemented using both COMSOL and a finite difference form. The finite difference formulation is re-formulated in a set of ODE's for direct implementation in an optimization software. In a third approach, a discrete control volume approach is developed and linearized for subsequent control of water injection to meet the desired moisture level. The key parameter to study the moisture content is the effective water saturation, S_e. Questions about system and valve opening time scale are answered from the saturation breakthrough curves. The overall results points towards methods for solving the problem and underlines a theoretical positive answer to the question posed. Further analysis using the models or actual experiments are needed to confirm the findings.

Contents

1 Introduction

Climate changes are causing a significant change in water distribution system with less precipitation in the summer and more in the winters (in general for Europe). Hence, research and development for crop's drainage mechanism gains more and more importance. Therefore, DLF-Trifolium, LKF Vandel, Nordic Seed, Copenhagen University, Aarhus University and Aalborg university are joining efforts to investigate crop drain within the research project: "Crop Innovation Denmark" which conspires to confirm/ validate following hypotheses:

- There are considerable genetic variation for rooting depth
- Rooting depth and drought tolerance are correlated

In order to investigate this, the company intends to set up a test facility where soil moisture level at a given depth can be controlled at any given point. The control should be realized by injection of water through strategically placed valves and sensors for moisture readings. This way it will be possible to investigate how plants behave at different water saturation levels.

The problem submitted to this ESGI workshop is quite open. To understand the system behavior a model is required. For this reason we explore three different approaches for modeling:

- Two-Phase flow: In general the problem is a case of air and water flow in a porous medium.
- Richards equation in 2D [1]: This equation is normally used to simulate water flow in unsaturated soil.
- Discretized flow model: In order to formulate the problem as a control problem a roughly discretized model is implemented for linearization.

The ultimate goal will be to formulate a (reduced order) model on which an optimal control algorithm can be implemented[4].

2 Problem formulation

How can one, by controlled water supply at various points, create a level surface of constant moisture that reflect the bottom surface of the field? The DLF Trifolium has not yet decided on any exact model for the test setup and they are considering several options for constructing a research facility. However, the overall idea is to create a test facility, where rows of different varieties will be planted in the longitude direction, see Figure 1.

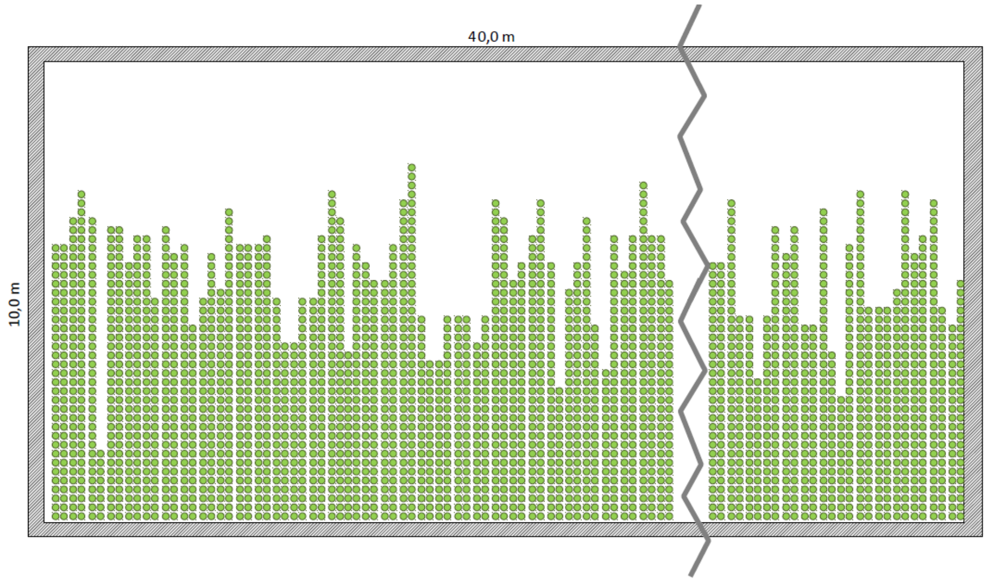


Figure 1: Test area seen from above

The size of the test field is from above 10×40m. Under this field, an idea is that distance from the surface to the moist soil will be different along the grave, such that the roots will have a greater and greater distance to the water. The DLF Trifolium has not specified the overall design of this slow decrease of water levels. Therefore, we have decided to consider two different model layouts.

2.1 Separated terrace model

Considering the problem geometry as a separated terrace model, we only need to model how the water behaves in one of the boxes, and then we can apply this to the rest. A separate terrace model can be seen in the Figure 2. The advantage of the separated terrace model is not only from a modeling point of view, but, also from an experimental point of view.

- The roots for plants at different depth will not get mixed/entangled.
- The separated terrace model allows for a larger sample of grass with the same distance to water. Hence, one will be able to say with more statistical precision with what root length grass can survive.
- If something breaks or goes wrong, for example if some bad water gets in, it will only effect a part of the experiment and does not ruin the whole.

- Another advantage, when using this model, is that it is quite easy to calculate the individual plant transpiration. Since we know for each depth the exact amount of water coming in through the valves and leaving through the drain, the rest must be water absorbed by the plants. This information together with the frequent pictures of the plants, taken at different light spectra, provides the water status in the individual plants. Water status can be used to asses which plant varieties are best at transpiration.

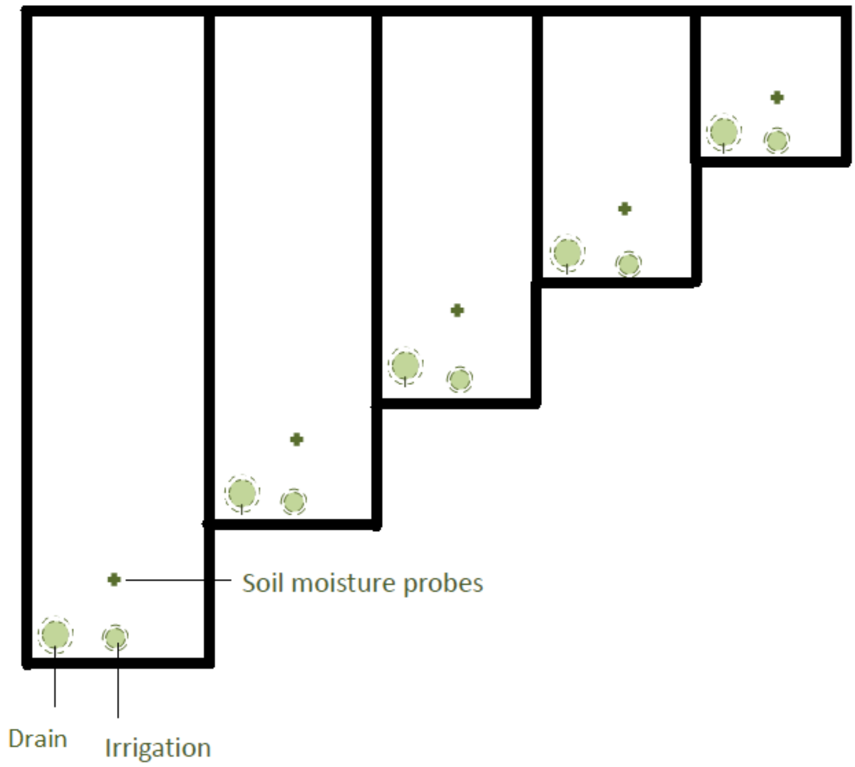


Figure 2: The separated terrace model.

2.2 Uniform area model

In the uniform model the system is considered as one grave with several valves and sensors. The right side of the grave is 0.5m high and the left-hand side is 3m. Width of the grave set to 10m results in a slope of 14 degrees of the bottom of the grave. The uniform area has a constant slope and is indicated in Figure 3.

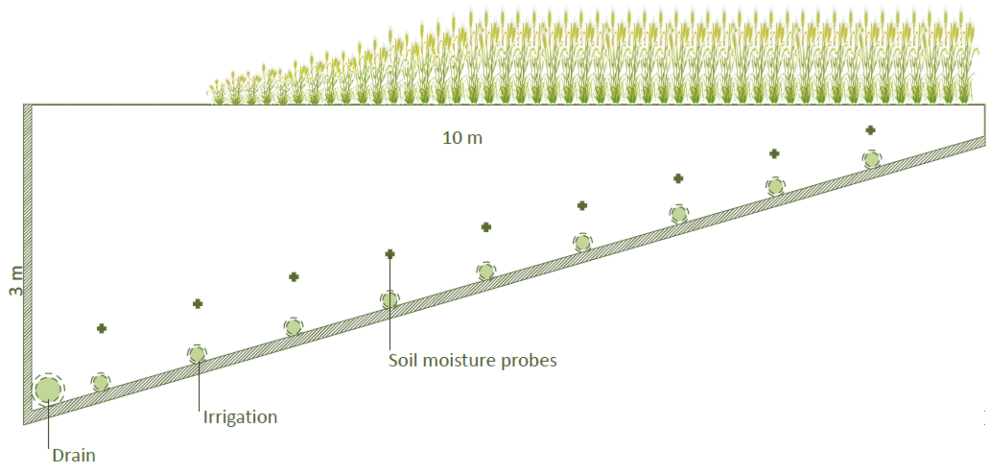


Figure 3: The uniform area model.

In both models a number of valves will be placed from which water can be added to the grave. Above each valves, a sensor is placed for saturation reading. If a sensor reads saturation that is below the lower threshold of saturation then the valve should turn on for adding water to attain the desired saturation level. A drain is placed at the left bottom to ensure that extra water is flowing down due to gravity and does not accumulate at the bottom of the grave.

The following constraints are given in both cases:

- There is a limit of 400 valves available and for each valves a sensor is assigned.
- The drainage can be opened or closed when extra water is required to drain out.
- We will tolerate a variation of 8% from the desired saturation in maximum 30 minutes.

3 Modeling approaches

3.1 Two-phase flow

The flow of water and air through soil can be described as a two-phase flow in a porous medium. The mass balance of air and water describe water saturation distribution in time and space. In general, on the complex geometry of porous media, the balance equations are partial differential equations, solvable by using the volume-averaging technique. This technique can be achieved by averaging these governing partial differential equations on a representative elementary volume (REV) of the porous medium. Mass balance equations are discretized in space and time within the context of the finite element method (FEM). In particular, a Galerkin's procedure is used for the discretization in space. The finite element implementation is based on a C++, Object Oriented Code.

3.1.1 Mathematical model

The grave in the research facility will consist of a uniform soil. Modeling of such two-phase flow in porous media requires the consideration of mass transfer between the liquid and gas phases by advection and diffusion processes. Between the grains of the soil, there will be either water or air if the saturation of one phase (water) grows, the saturation of other phase (air) decreases and vice versa. Water and air flow in the porous media can be described by two mass balance equations:

$$\frac{\partial \left(nS^l\rho^l\right)}{\partial t} + \nabla \cdot \left(n\mathbf{v}^l\rho^l S^l\right) = n\rho^l S^l Q^l \tag{3.1}$$

$$\frac{\partial \left(\rho^g n S^g\right)}{\partial t} + \nabla \cdot \left(n\mathbf{v}^g\rho^g S^g\right) = n\rho^g S^g Q^g \tag{3.2}$$

Here, superscript l refers to the liquid (water) and g to the gas (air) phases, respectively. n is porosity, S^l ($S^l + S^g$=1) is water saturation, ρ is density, and Q is sources/ sink term.

We define the fluid volume, as being the volume of air plus the volume of water $v^f = v^g + v^l$. Here, gas phase and liquid phase pressures are linked through capillary pressure, $p^c = p^g - p^l$. And saturation is obtained as, $S^l = \left(\frac{p_d}{p^c}\right)^m$ for $m = 2$ and $p_d = 5000$ Pa. Porosity, n, is given by the free volume divided by the total volume. The total volume is the volume of the soil, water and air $n = \frac{v^f}{v}$. The volume of water divided by the fluid volume gives the water saturation. $S^l = \frac{v^l}{v^f}$. The term, Q^l and Q^g, is a measure of the amount of the phase added or extracted. In order to find the velocity of the air and the water, we utilize Darcy's law. It quantifies the flow rate of fluid in a porous medium. We write the flow rate for water and air as follows:

$$q^l = n\mathbf{v}^l = -\frac{\mathbf{K}k_r^l}{\mu^l}\left(\nabla p^l - \rho^l\mathbf{g}\right) \tag{3.3}$$

$$q^g = n\mathbf{v}^g = -\frac{\mathbf{K}k_r^g}{\mu^g}\left(\nabla p^g - \rho^g\mathbf{g}\right) \tag{3.4}$$

In the Darcy flow equations, $\mathbf{K}$ is intrinsic permeability measure the ability if fluid flow in porous media. And k_r^l and k_r^g are liquid and gas relative permeability. The relative permeability describes relative flow of one phase with respect to other. Here, μ is viscosity, p is pressure, and g is gravity vector. The relative permeability is depending on their respective saturation. These are

$$k_r^l = S_e^2 \left(1 - S_e\right)^2 ; k_r^g = S_e^4. \tag{3.5}$$

and

$$S_e = \frac{S^l - S_0}{1 - S_0} \tag{3.6}$$

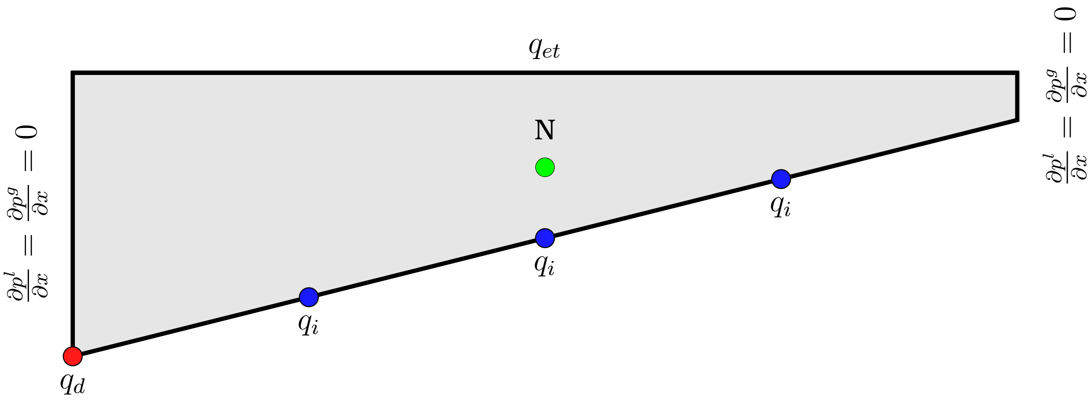
A certain value of water saturation are always associated with solid matrix. This cannot drain out and is called the residual saturation, S_0.

3.1.2 Model description

The problem of water levels in a root screening facility is modeled. The model geometry is two-dimensional as shown below. This system is considered as porous media. The pores of the solid matrix are filled with water and air at a fixed temperature ($T = 298.15K$). In such systems the moisture level is represented by the water saturation, S_{e-}. Initially the moister level of the system is 50%, i.e., $S_{e0} = 0.5$ at time $t = 0$. System properties and material parameters used in the numerical simulation (porous medium as well as of fluid and solid phases) are summarized in Table 1.

The goal of this model is to estimate the time when (a) water saturation declined to a certain low value due to vaporization, transpiration and drainage of extra water; (b) water saturation attained to the desired high value S_{e+} so crops does not over saturated due to water injection. The hypothesis is that water saturation breakthrough curve at point N $(5, -0.5m)$ can suggest an optimal time for water injection.

The amount of lost water if the saturation goes to lowest value (S_{e-}) from initial value (S_{e0}) is $A\rho_w(S_{e0} - S_{e-})$ kg/m. Hence, the vaporization rate from the top of domain is $q_{et} = \frac{A\rho_w(S_{e0}-S_{e-})}{Lt_c} = 7.23 \cdot 10^{-3}\mathrm{kgm}^{-2}\mathrm{s}^{-1}$. To make things simple, vaporization and transpiration rates are combined in (q_{et}). Here, $A = 17.5\mathrm{m}^2$ is area, $\rho_w = 1000\mathrm{kgm}^{-3}$ is the water density, $L = 10\mathrm{m}$ is length of domain at top and $t_c = 240,000\mathrm{s}$ is time for one period. Mass of water required to saturate the whole system from S_{e-} to S_{e+} is: $A\rho_w(S_{e+} - S_{e-})$ kg/m. Hence, the water injection rate from each of three valves is $q_i = \frac{A\rho_w(S_{e+}-S_{e-})}{3t_i} = 14.41 \cdot 10^{-3}\mathrm{kgm}^{-1}\mathrm{s}^{-1}$ for a period of $t_i = 60,000\mathrm{s}$. To overcome the gravity effect, extra water is allowed to drain out by assigning drainage rate $q_d = \frac{3q_i t_i - q_{et}Lt_c}{t_c}$.



Property	Symbol	Value	Unit
Area	A	m^2	17.5
Air viscosity	μ^a	Pa.s	1.0×10^{-5}
Water viscosity	μ^w	Pa.s	1.0×10^{-3}
Air density	ρ^a	$\mathrm{kg.m}^{-3}$	1.0
Water density	ρ^w	$\mathrm{kg.m}^{-3}$	1.0×10^3
Permeability	$\mathbf{K}$	m^2	1.0×10^{-10}
Porosity	n	$--$	3.0×10^{-1}
Residual saturation of water	S_{rw}	$--$	0
Residual saturation of air	S_{ra}	$--$	0
Entry pressure	p_d	Pa	5.0×10^3
Soil distribution index	λ	$--$	2.0
Capillary pressure	$p^c(S_{eff})$	Pa	Brooks-Corey model
Relative permeability	$\kappa_{rel}(S_{eff})$	$--$	Brooks-Corey model
Specific storage	S_s	$\mathrm{kg.Pa}^{-1}$	1.5×10^{-8}

Table 1: Material parameters used in the numerical simulation.

3.1.3 Numerical simulation

The two-dimensional problem domain is discretized into 1734 triangular elements. An self-adaptive time step control method is used for time stepping. In this method an optimal size for the time step is automatically calculated. For solution of the effective water saturation S_e, numerical simulation has been performed. Material parameters of the porous medium are presented in Table 1.

In Figure 4, iso-saturation contours are presented at $t = 5^{th}$ and $t = 15^{th}$ hour of the water injection. Figure 5 shows breakthrough curve of effective water saturation, S_w, at the point N. The trend of the breakthrough curve suggests that a periodic steady state is attained. Periodic nature of the curve helps to identify different times such that (a) when saturation reaches a certain lower threshold value ($S_e = 0.4$) due to vaporization and transpiration; (b) when the saturation level attains a desired value ($S_e = 0.55$) from the lowest threshold value ($S_e = 0.4$) due to water injection from the valves. In this case study, S_e decreases to $S_e = 0.4$ within 50 hours. And, to reach the desired level of water saturation water injection is required for 16 hours at the considered rate. The results are based on the rates (injection, vaporization, drainage) and parameters that are given in model description subsection.

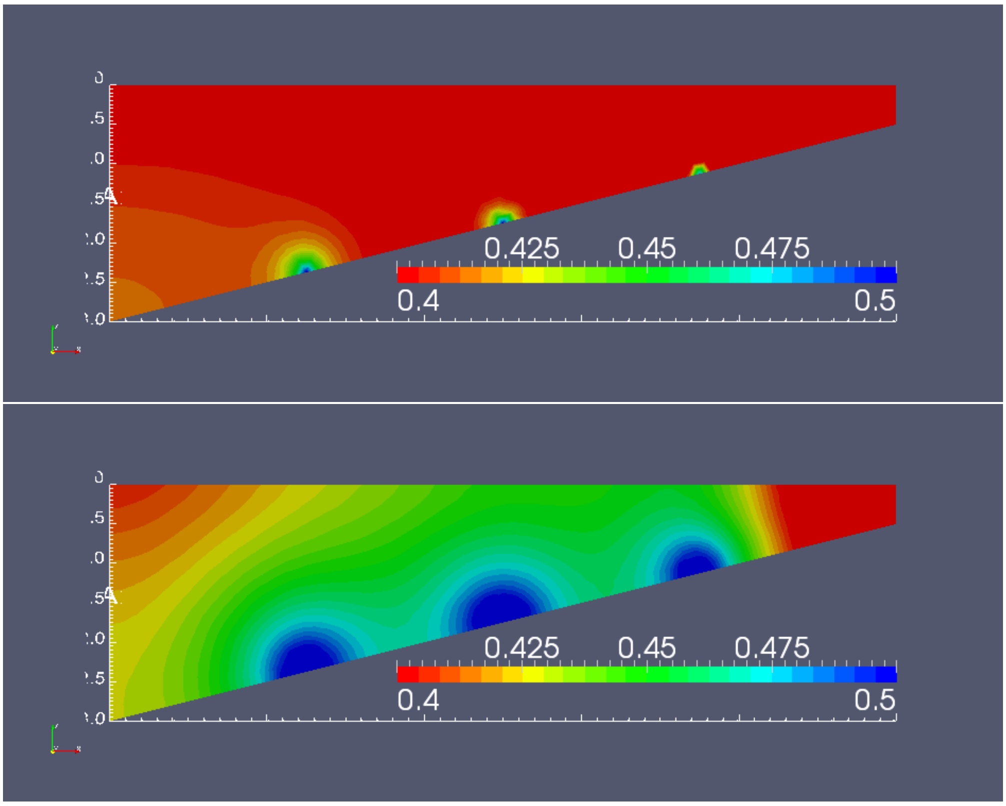


Figure 4: Distribution of the effective water saturation over the problem domain at $t = 55$ (top) and $t = 65$ hour of simulation.

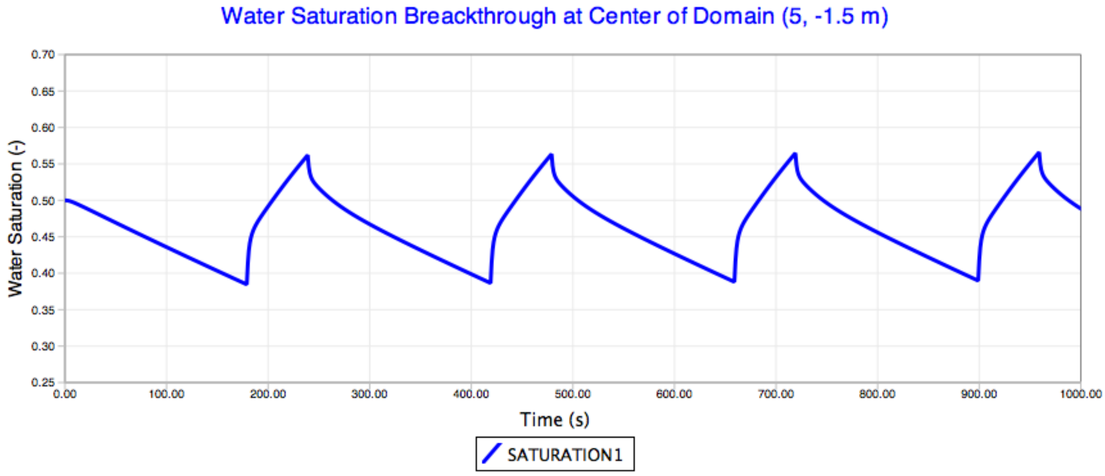


Figure 5: Breakthrough curve of effective water saturation at point O.

3.2 Richards flow

When soil is homogeneous and isotropic porous medium the movement of the water can be described by the Richards equation. This equation is derived based on the further assumption that air pressure is constant in the two-phase flow.

The equation is given as follows, as given in [1] whith extra source terms:

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[K(h)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right] - \frac{\partial K(h)}{\partial z} - S_w + S_v - S_d \tag{3.7}$$

h is the hydraulic head of the water.

$K(h)$ is the hydraulic conductivity of the porous media depends on the intrinsic permeability, saturation, density and viscosity of the fluid.

$C(h)$ is the specific water capacity, i.e. the rate of change of water content with change in water pressure. Thus, this is defined as $\frac{\partial \theta}{\partial h}$. Here, θ is liquid saturation defined as: $\theta = \frac{v_l}{v_t}$. This results in a left-hand side being $\frac{\partial \theta}{\partial t}$

S_w describes the root water uptake, S_v describes flow of water from the valve and S_d describes the flow of water out the drain.

S_w is defined as

$$S_w(x,z) = \alpha(h) \cdot S_p(x,z) \tag{3.8}$$

S_p describes the potential root water uptake and α is a reduction factor due to water stress. α depends on the pressure of the water as can be seen in the Figure 6

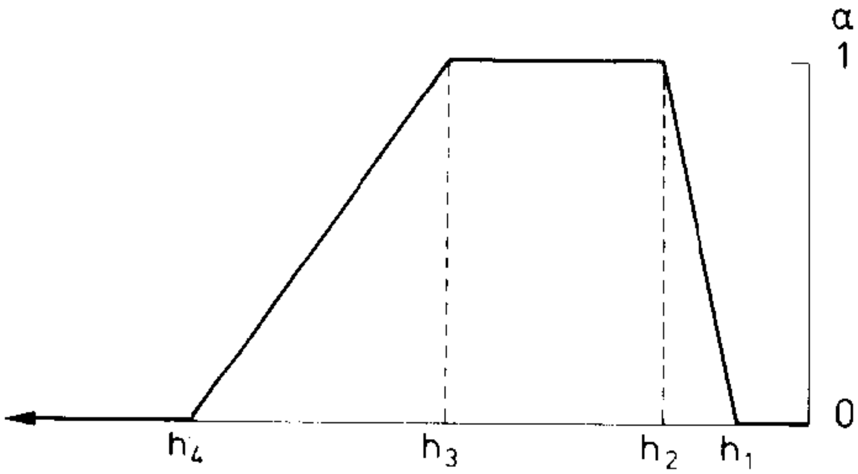


Figure 6: α's dependence on the water pressure, h, [3]

If the soil is wetter than a certain 'anaerobiosis point', h_1, water uptake is 0 and if it is drier than the wilting point, h_4, it is zero. Between h_2 and h_3 it is at its maximum uptake.

The water content $\theta(h)$ and the hydraulic conductivity, $K(h)$, are described in the Brook-Corey relation.

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = \left(\frac{h_a}{h}\right)^n \tag{3.9}$$

$$K(h) = K_s \cdot \left(\frac{h_a}{h}\right)^m \tag{3.10}$$

Here θ_s is the saturated moisture content, which is equal to the porosity, i.e. θ_s is defined as the free volume (total volume subtracted volume of the soil) divided by total volume, $\theta_s \frac{v_f}{v_t}$. θ_r is the residual moisture content i.e. the minimum content of water in the soil. Here, $C(h)$ becomes zero. h_a is the air entry value and K_s is the saturated hydraulic conductivity. n and m are parameters related by $m = 2 + 3n$.

The finite difference method for the Richards equation has been implemented in `python` in a single box.

The function α from equation 3.8 has been approximated by a smooth function defined as

$$\alpha(h) = \frac{\tanh(A \cdot h + B) + 1}{2} \cdot \frac{\tanh(C \cdot h + D) + 1}{2} \tag{3.11}$$

with

$$A = -\frac{3.8}{h_1 - h_2} \tag{3.12}$$

$$B = -A \cdot h_1 \tag{3.13}$$

$$C = \frac{3.8}{h_3 - h_4} \tag{3.14}$$

$$D = 3.8 \cdot \left(1 - \frac{h_3}{h_3 - h_4}\right) \tag{3.15}$$

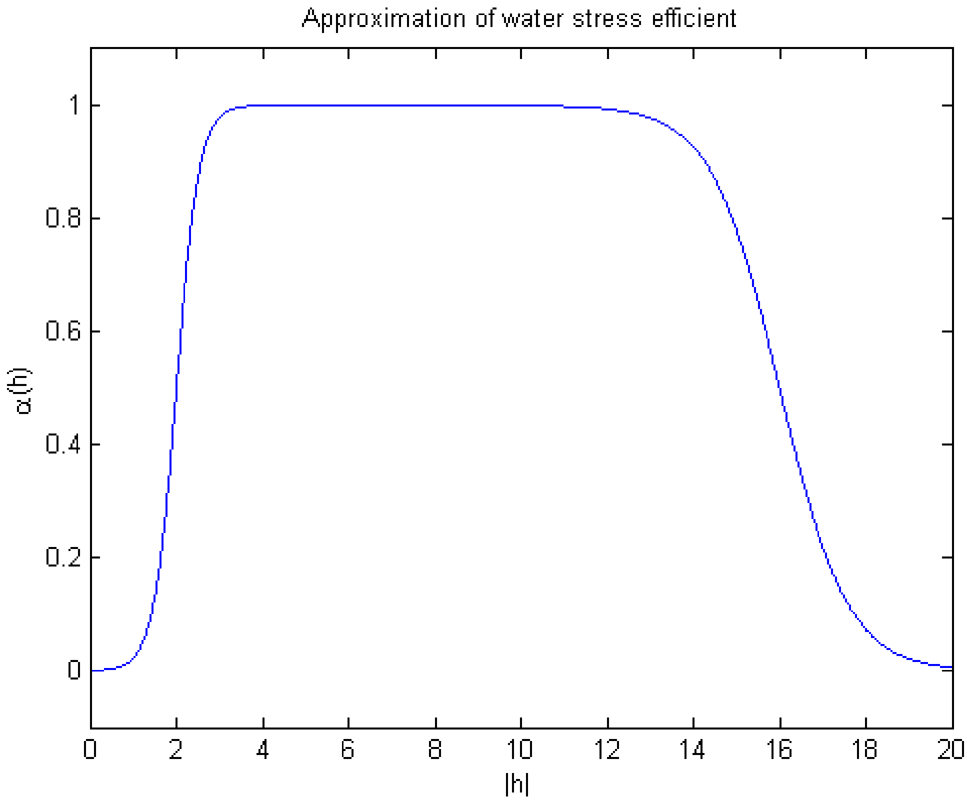


Figure 7: The smooth approximation of α

α values for different soil types can be seen in the Figure 8. The loam value has been used in the implementation as we assume ideal soil for gardening and agricultural use.

Soil type	Brook-Corey model parameters					
	θ_s	θ_r	K_s (m s^{-1})	h_a (m)	n	m
Sand	0.417	0.02	5.83×10^{-5}	–0.0726	0.592	3.776
Loam	0.434	0.027	3.67×10^{-6}	–0.1115	0.22	2.66
Clay	0.385	0.09	1.67×10^{-7}	–0.373	0.131	2.393

Figure 8: Parameter values for the Brook-Corey model, [1]

The initial condition is set to

$$h(x,z,0) = h_0(x,z) \quad \text{for} \quad 0 \leq x \leq X, \quad 0 \leq z \leq Z \tag{3.16}$$

On the left and right boundary, following conditions are applied

$$\frac{\partial h}{\partial x} = 0 \quad \text{for} \quad x = 0, X \quad \text{and} \quad 0 \leq z \leq Z \tag{3.17}$$

On the upper boundary, it is assumed that the pressure gradient is zero, i.e. no flow. The moisture contents is described by S_w .

$$\frac{\partial h}{\partial z} = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad 0 \leq x \leq X \tag{3.18}$$

In figure 9 the results from a finite difference inplementation of the Richards equation is presented.

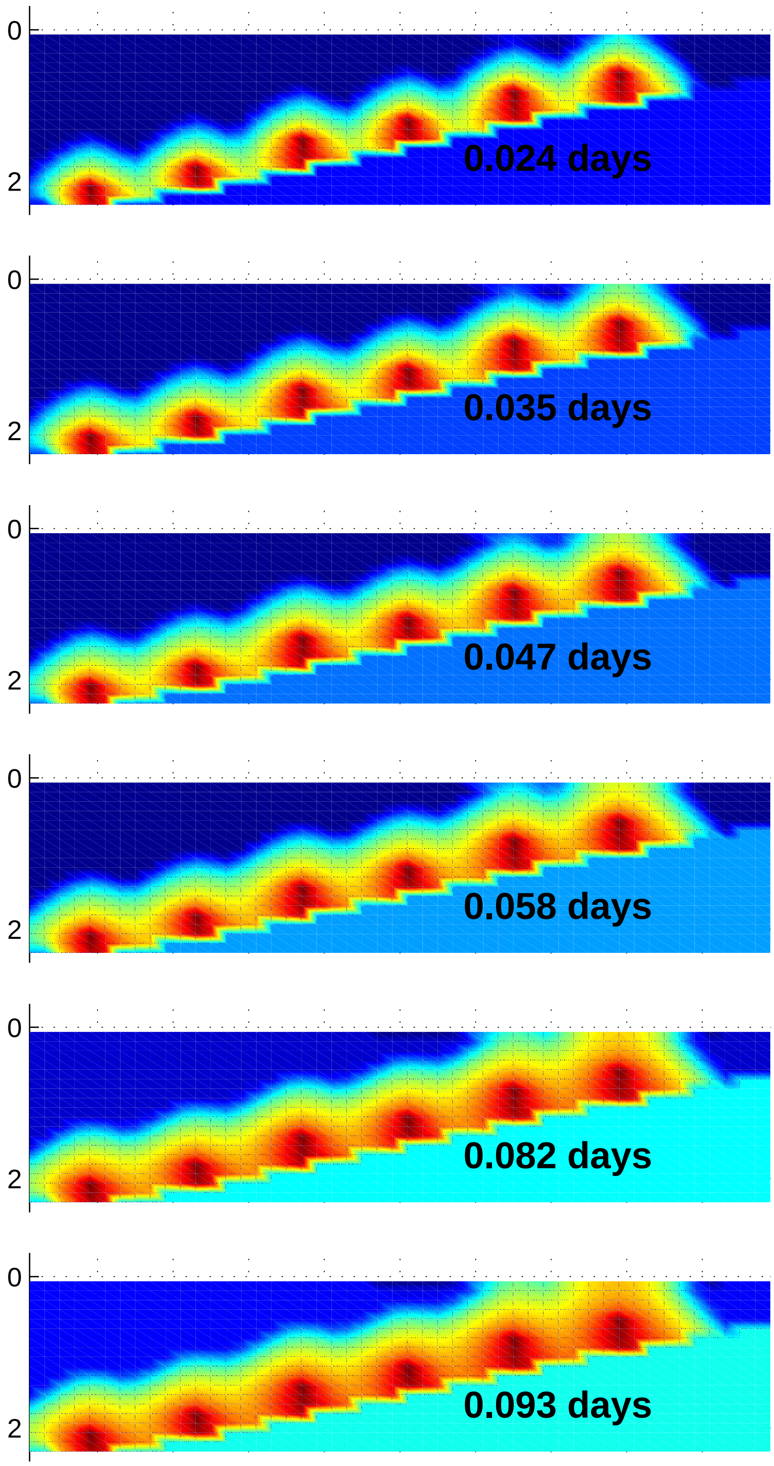


Figure 9: 2D Richards with 6 valves and 1 drain

3.2.1 Comsol implementation of the Richards equation

The Richards equation as described above was also implemented in COMSOL. In fact COMSOL has a predifened physics module for solving these kind of problems. We start out by modeling the area using 4 valves and 4 sensor, and one drain. The level of saturation is initially 1 everywhere. We then start the drain, and the saturation begins to drop. A sketch of the area can be seen in Figure 10.

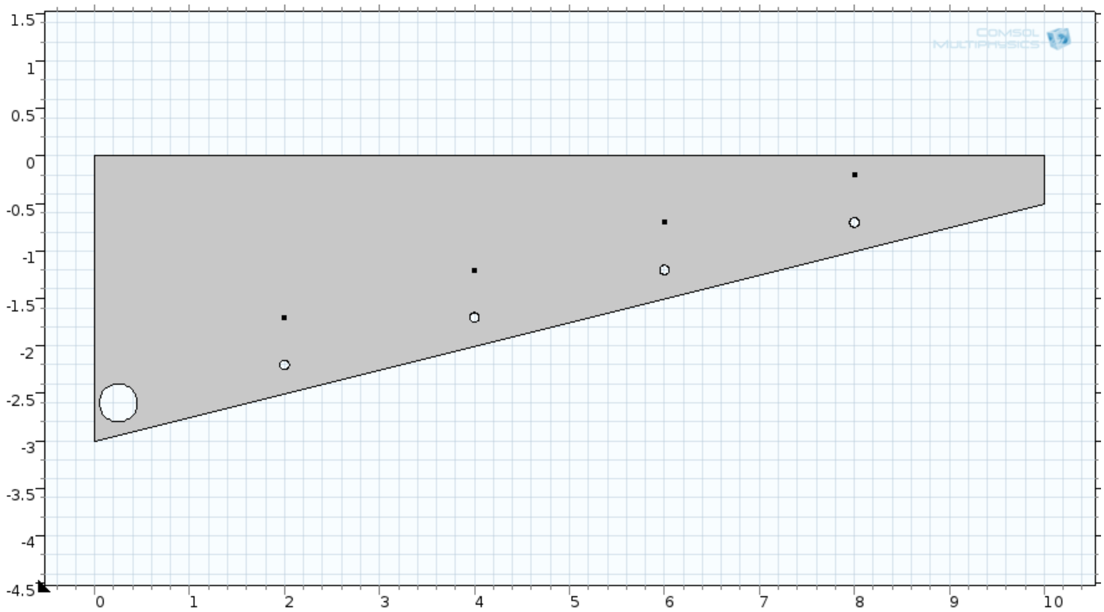


Figure 10: Sketch of the basic model. The valves are numbered from left towards right, meaning that the leftmost valve is number 1 and the rightmost valve is number 4.

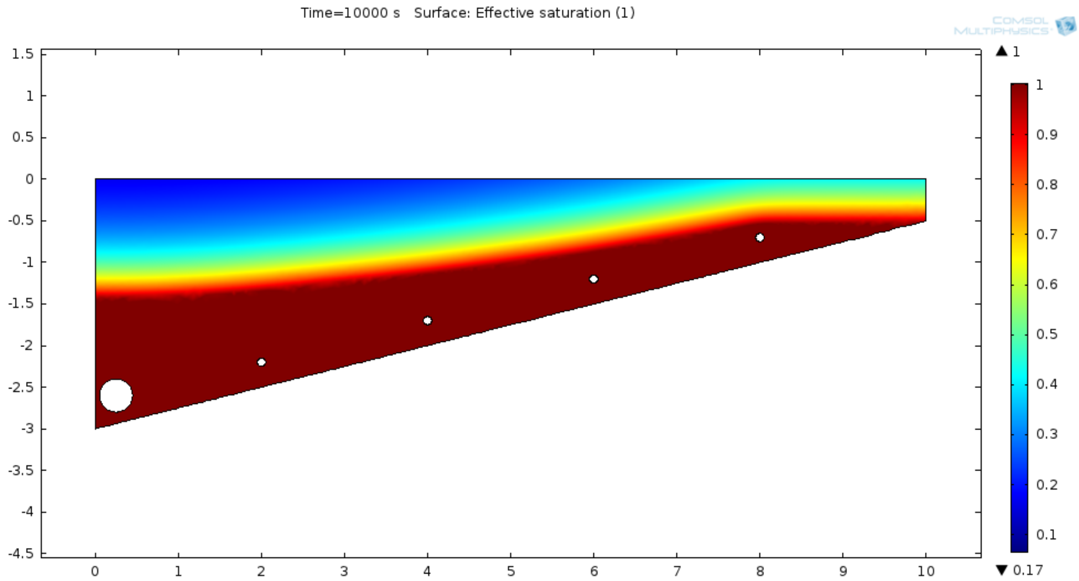


Figure 11: In this figure we have put in a moisture lines, and we then turn on valve number 4.

After the saturation has been dropping below 0.8, we turn on the water injection from the valve number 4 and solve the steady state solution. We can see that the moisture level around valve number 4 rises naturally. In the Figure 12, one can see how water gets injected. At the beginning valve is closed, then, a large injection is made, hereafter the saturation level begins to stabilize.

We solve the problem by using a PI controller on sensor 4. In Figure 13, one can see how the saturation changes with water injection. At first, the saturation is 1, then due to vaporization saturation fall to our moisture target (0.8) at point 4.

By varying the parameters in the PI controller, it is possible to control the fluctuations in a better way. However, the reader should consider this as a first cut on a solution. This suggests possibility of adjustment in the water levels. However, without optimum parameters it does not make sense to optimize solution.

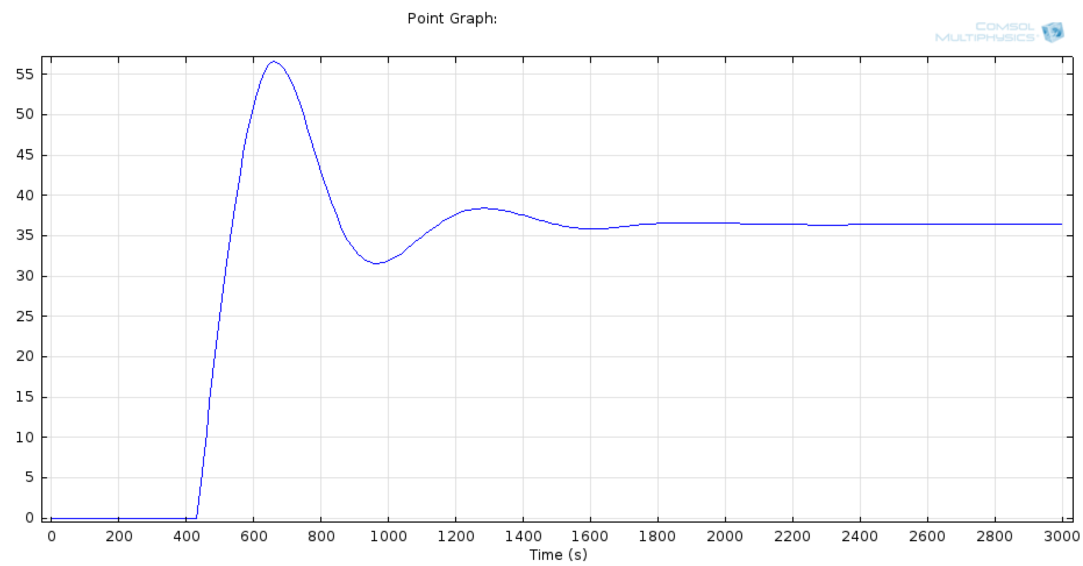


Figure 12: Graph of the injection of water provided by valve 4.

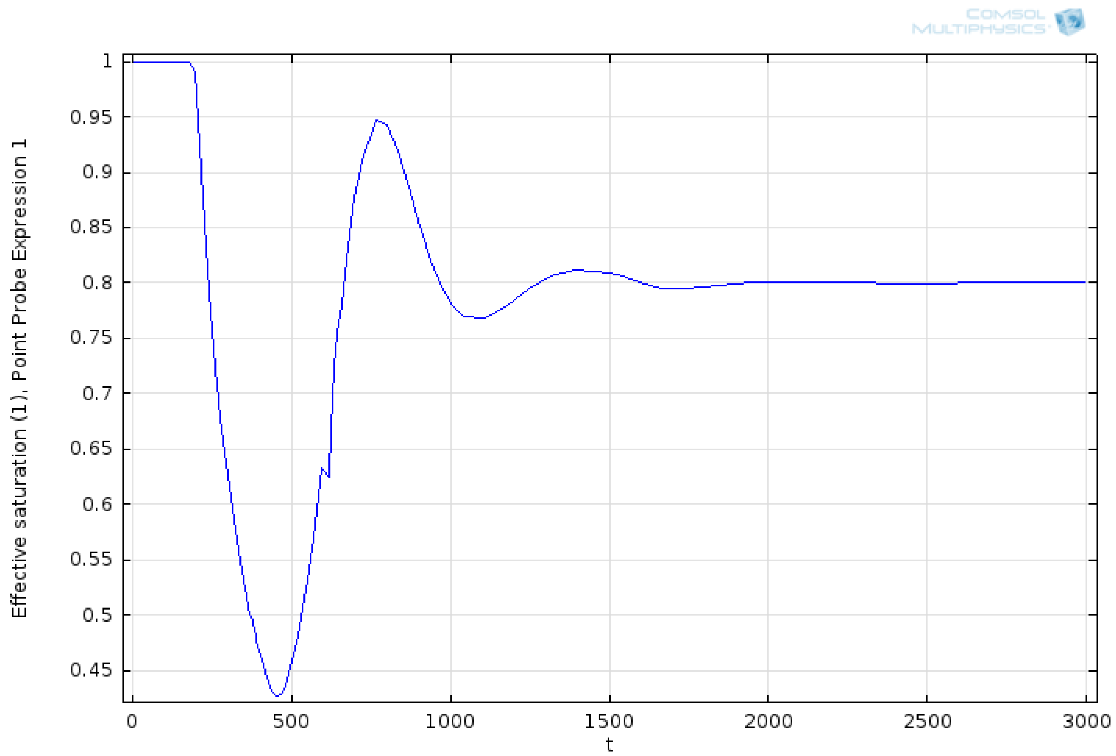


Figure 13: Graph of the saturations before and after the injection, measured by sensor 4.

Looking at what happens at the other valves, see Figure 14, we see that valve number 3 also experiences big changes in the saturation. Whereas, the saturation around sensor 1 and 2 barely changes.

The overall moisture distribution at 3000 s can be seen in 15. The moisture level at sensor 4 is now 0.8. The reader should know that the sensors are not drawn in this plot, but, by looking at the Figure 10 one can get an idea of where the sensors are located.

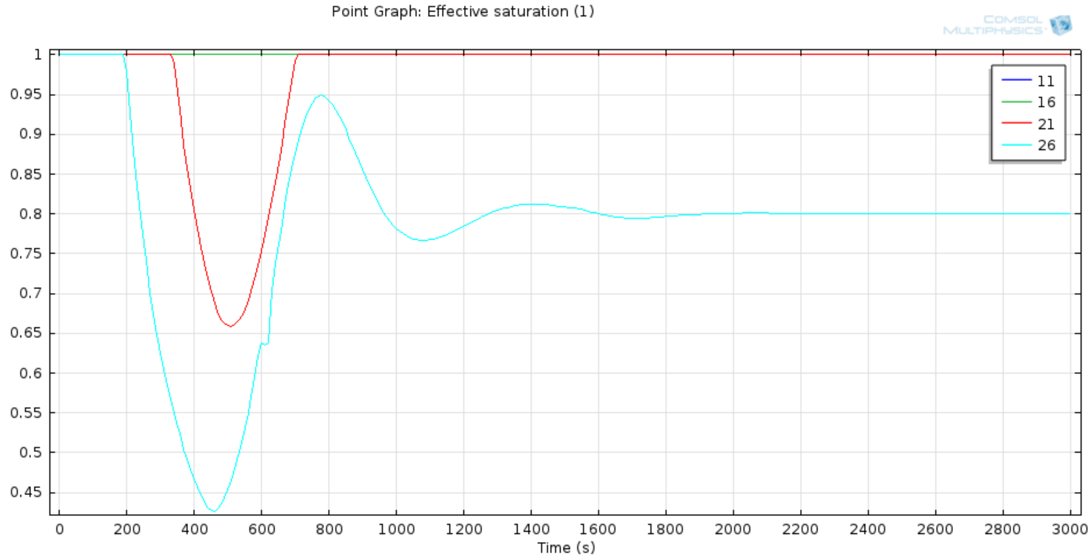


Figure 14: Shows the moisture level at the other valves.

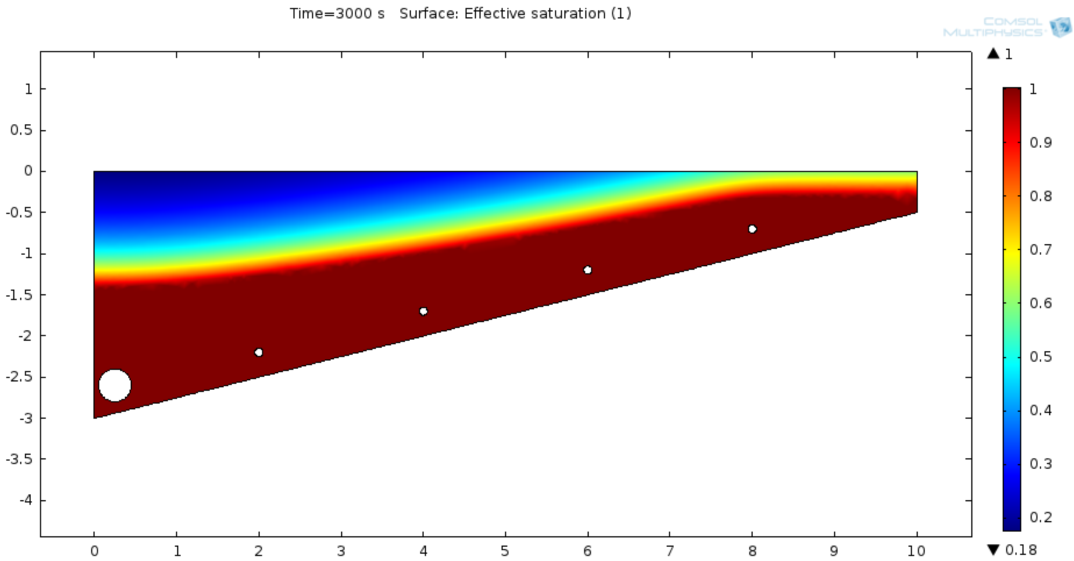


Figure 15: Moisture plot after 3000s.

3.3 Discretized flow

Another way to consider the problem is to discretize the area in question, see Figure 16.

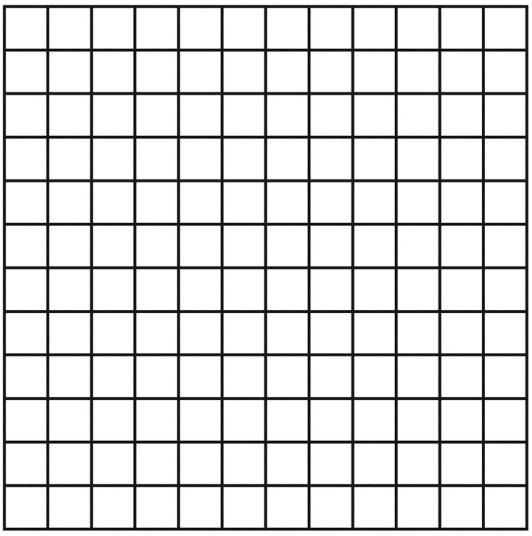


Figure 16: Discretized area

We describe the structure of the area by assigning a vector of 8 bits to every cell – $(b_k^1, b_k^2, \ldots, b_k^8)$, where:

- k - the index of a cell,
- b_k^1 - can water flow between this cell and the cell above it,
- b_k^2 - can water flow between this cell and the cell to the right,
- b_k^3 - can water flow between this cell and the cell below it,
- b_k^4 - can water flow between this cell and the cell to the left,
- b_k^5 - is there a sensor in the cell,
- b_k^6 - is there a valve in the cell,
- b_k^7 - is there a drain in the cell,
- b_k^8 - is it solid (0 means it is air).

In each cell point, we consider only the effect of its neighbors and we direct the flow to the left and downward. This does not mean that it can only flow these ways. The value of the flow will be negative if the flow is in the opposite direction.

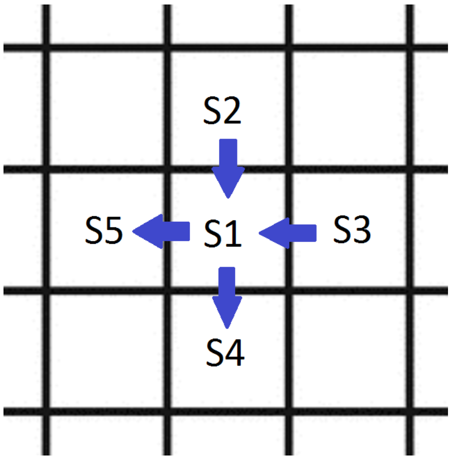


Figure 17: One grid point and its neighbours

Taking a look at Figure 17 we can see that S_1 has four neighbors affecting it. This gives an equation

$$\dot{S}_1 = q_{2-1} + q_{3-1} - q_{1-4} - q_{1-5} + u_1 \tag{3.19}$$

where q_{i-j} describes the flow from i to j. u_1 is a source/sink. This will be positive if we have a valve in the grid point, negative if we have a drain and 0 if it has neither.

The flow in the example is described as follows:

$$q_{2-1} = -K_E^* S_1, \tag{3.20}$$

$$q_{3-1} = K_S^* \left(S_3 - S_1\right), \tag{3.21}$$

$$q_{1-4} = K_S^* \left(S_1 - S_4\right) + \rho S_1, \tag{3.22}$$

and

$$q_{1-5} = K_S^* \left(S_1 - S_5\right). \tag{3.23}$$

In this example we assumed that cell 2 represent air and the other were solid. Therefore we used K_E^* for the transfer between cells 2 and 1, and K_S^* for the transfer between solid cells. There is also a constant ρ^* related to the flow due to gravity.

Numerical values for K_S^*, K_E^*, ρ^* are dependent on the distance between center of the cells, porosity, viscosity of the ground, and the saturation itself. The overall differential equation for the system will be

$$\dot{s}(t) = As(t) + Bu(t) \tag{3.24}$$

and we want to minimize

$$J = \sum_k |s_k - s_k^t|^2 \tag{3.25}$$

We assume here that there are $I \cdot J$ cells (I rows, J columns) in the grid and we enumerate them using the following rule: first row starting from the left, next row starting from the left, and so on. Here, s_k is the saturation in grid point k which corresponds to two-dimensional indices according to the formula $k = i \cdot J + j$, where i is the row number and j is the column number. s_k^t is the desired saturation value i.e. the value we want to attain.

Matrix $A \in M_{I \cdot J, I \cdot J}$

This model assumes that we know that value everywhere, i.e. we have sensors in every point. In fact we do not have it.

4 Control theory

In control theory one is concerned with the following minimization problem

$$\min_{x(\cdot),u(\cdot)} \int_0^T L(t, x(t), u(t), p)\mathrm{d}t + E(x(T)) \tag{4.1}$$

subject to the following constraints

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T], \tag{4.2}$$

$$x(0) = x_0, \tag{4.3}$$

$$h(x(t), u(t)) \leq 0, \quad t \in [0, T], \tag{4.4}$$

$$r(x(T)) \leq 0. \tag{4.5}$$

Here, we have a Lagrange term

$$\int_0^T L(t, x(t), u(t), p)\mathrm{d}t \tag{4.6}$$

and a Mayer term

$$E(T, x(T), p) \tag{4.7}$$

In this case the minimization will be to minimize the time in which we are not at the desired saturation. Here, we have the state function, x, which in this case could be the saturation and the control function, u, describing the values we control, e.g. injection of water from the valves.

Then we will have some condition on the state and control function. This could be an upper and lower bounds. The last condition will be on the state function at the end time, T.

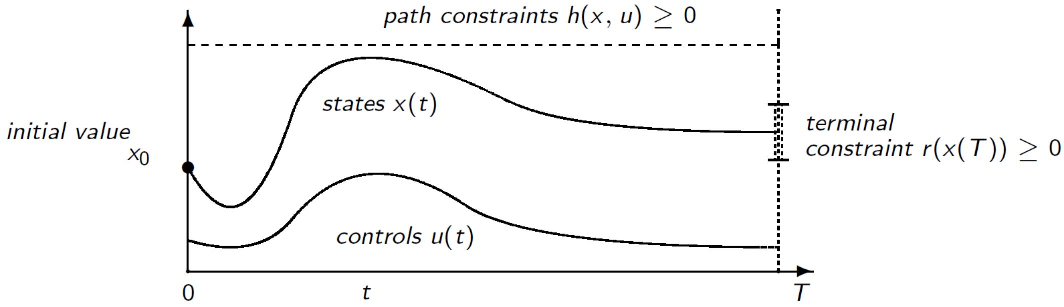


Figure 18: State and control function dependence and conditions, [4]

In Figure 18 the problem is illustrated. We start the state function from an initial state, x_0, then adjusting the control function the state function react. The constraint on the state and control can be seen as an upper bound. The constraint on the end time is seen as a constraint that x need to be in a certain interval which corresponds to the fact that we want to be within $\pm 8\%$ of a certain moisture level.

4.1 Model predictive control

The MPC approach was chosen in order to design a controller. There are two possible ways of using this method. The most complicated, but also very reliable, is based on nonlinear differential equations describing the model. However, there are some simplifications, e.g. using linearization, which provide satisfactory results. The linearization has also another advantage - it allows very often to verify the stability of the system, which is important for controller design. The nonlinear system can be written as

$$\dot{x}(t) = f(x(t), u(t)) \tag{4.8}$$

where $x(t)$ denotes the vector of states variables and $u(t)$ is the control function (also a vector).

The analyzed systems consists of a vector $x(t)$ of nine variables which denotes the saturation in "cells" and the control function $u(t)$ that contains two variables. The justification for such a model can be found in figure 19.

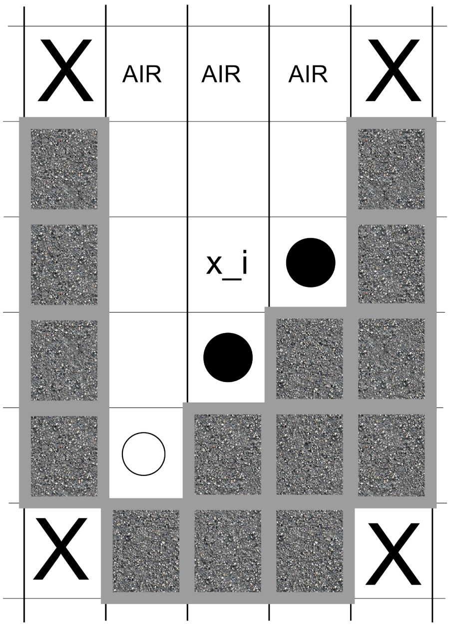


Figure 19: Model used to design MP controller.

The first step for linearization is to find the equilibrium/the operating point in which the linear approximation will be calculated. Using the fact that the operating point (x_p, u_p) is a constant solution to equations from 3.3 one has the following equation to solve

$$0 = \dot{x}_p = f(x_p, u_p) \tag{4.9}$$

In most cases, the operating point is given arbitrarily. In this case only some information on x_p was given. Therefore, it was necessary to find the appropriate control values. The MATLAB function `lsqnonlin` was used for that purpose. Hence, the problem of finding a solution for a system of algebraic nonlinear equations was transformed into an optimization problem

$$\min_x \|f(x)\|_2^2. \tag{4.10}$$

The results are the following

$$x_p = \begin{bmatrix} 0.47 & 0.91 & 1.13 & 0.78 & 1.16 & 1.4 & 0.811.2 & 0.3162 \end{bmatrix}^T, \, u_p = \begin{bmatrix} 1.5 & 0.5 \end{bmatrix}^T.$$

The method of linearization can be found e.g. in [6]. It is based on Taylor series expansion and allows writing the equations of the system in the "classical" form of control theory

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)\end{aligned} \tag{4.11}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are constant matrices. In this case, the matrices are

$$\mathbf{A} = \begin{bmatrix} -0.66 & 0.78 & 0 & 0.43 & 0 & 0 & 0 & 0 & 0 \\ 0.21 & -2.31 & 1.35 & 0 & 1.53 & 0 & 0 & 0 & 0 \\ 0 & 0.62 & -3.08 & 0 & 0 & 3.17 & 0 & 0 & 0 \\ 0.11 & 0 & 0 & -1.45 & 1.72 & 0 & 0.32 & 0 & 0 \\ 0 & 0.66 & 0 & 0.56 & -7.7 & 6.53 & 0 & 1.55 & 0 \\ 0 & 0 & 1.47 & 0 & 2.81 & -9.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.28 & 0 & 0 & -1.74 & 1.95 & 0.14 \\ 0 & 0 & 0 & 0 & 1.37 & 0 & 0.62 & -3.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.55 & 0 & -0.39 \end{bmatrix} \tag{4.12}$$

and

$$\mathbf{B}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.13}$$

5 Discussion and future work

5.1 Future work

Some ideas for future work evolved:

- We suggest use of a sprinkles connected to the valves, as this helps to spread out the water and keep the desired moisture level.
- There are several firms making sub-surface irrigation systems, however , most of them are only go down to a depth of 0.5 m. It might be a very good idea to make contact with people from these firms, as they have required practical experience in this area. They may improve the existing sub-surface irrigation systems to start it working at lower depths.
- We did not considerer the longitude case. However, we believe that as the current problem have been solved then it is simple to solve with considering longitude case.
- Furthermore, we strongly urge the Trifolium to go with the terrace model. As terrace model is an easy controlled problem and it has some experimental advantage over the continuous model.

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